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Robust Model Predictive Control of a Nonlinear System with Known Scheduling Variable and Uncertain Gain

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Abstract: Robust model predictive control (RMPC) of a class of nonlinear systems is considered in this paper. We will use Linear Parameter Varying (LPV) model of the nonlinear system. By taking the advantage of having future values of the scheduling variable, we will simplify state prediction. Because of the special structure of the problem, uncertainty is only in the B matrix (gain) of the state space model. Therefore by taking advantage of this structure, we formulate a tractable minimax optimization problem to solve robust model predictive control problem. Wind turbine is chosen as the case study and we choose wind speed as the scheduling variable. Wind speed is measurable ahead of the turbine, therefore the scheduling variable is known for the entire prediction horizon.

Keywords: Robust model predictive control, linear parameter varying, nonlinear systems, wind turbines, LIDAR measurements.

1. INTRODUCTION

Model predictive control (MPC) has been an active area of research and has been successfully applied on different applications in the last decades (Qin and Badgwell [1996]). The reason for its success is its straightforward ability to handle constraints. Moreover it can employ feedforward measurements in its formulation and can easily be extended to MIMO systems. However the main drawback of MPC was its on-line computational complexity which kept its application to systems with relatively slow dynamics for a while. Fortunately with the rapid progress of fast computations, off-line computations using multi-parametric programming (Baotic [2005]) and dedicated algorithms and hardware, its applications have been extended to even very fast dynamical systems such as DC-DC converters (Geyer [2005]). Basically MPC uses a *model* of the plant to *predict* its future behavior in order to compute appropriate control signals to *control* outputs/states of the plant. To do so, at each sample time MPC uses the current measurement of outputs and solves an optimization problem. The result of the optimization problem is a sequence of control inputs of which only the first element is applied to the plant and the procedure is repeated at the next sample time with new measurements (Maciejowski [2002]). This approach is called receding horizon control. Therefore basic elements of MPC are: a model of the plant to predict its future, a cost function which reflects control objectives, constraints on inputs and states/outputs, an optimization algorithm

and the receding horizon principle. Depending on the type of the model, the control problem is called linear MPC, hybrid MPC, nonlinear MPC etc. Nonlinear MPC is normally computationally very expensive and generally there is no guarantee that the solution of the optimization problem is a global optimum. In this work we extend the idea of linear MPC using linear parameter varying (LPV) systems to formulate a tractable predictive control of nonlinear systems. MPC problem of LPV systems has been considered in (Casavola et al. [1999]) and min-max MPC of LPV systems has been addressed in (Casavola et al. [2003]), however in this work we use future values of the scheduling variable to simplify the optimization problem. To do so, we use future values of a disturbance to the system that acts as a scheduling variable in the model. However there are some assumptions that restrict our solution to a specific class of problems. The scheduling variable is assumed to be known for the entire prediction horizon. And the nonlinear dynamics of the system is determined by the scheduling variable.

2. PROPOSED METHOD

Generally the nonlinear dynamics of a plant could be modeled as the following difference equation:

$$x_{k+1} = f(x_k, u_k) \quad (1)$$

With x_k and u_k as states and inputs respectively. Using the nonlinear model, the nonlinear MPC problem could be formulated as:

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$$\begin{aligned}
\min_u \quad & p(x_N) + \sum_{i=0}^{N-1} q(x_{k+i|k}, u_{k+i|k}) \quad (2) \\
\text{Subject to} \quad & x_{k+1} = f(x_k, u_k) \quad (3) \\
& u_{k+i|k} \in \mathbb{U} \quad (4) \\
& \hat{x}_{k+i|k} \in \mathbb{X} \quad (5)
\end{aligned}$$

Where $p(x_N)$ and $q(x_{k+i|k}, u_{k+i|k})$ are called terminal cost and stage cost respectively and are assumed to be positive definite. \mathbb{U} and \mathbb{X} show the set of acceptable inputs and states. As it was mentioned because of the nonlinear model, this problem is computationally too expensive. One way to avoid this problem is to linearize around an equilibrium point of the system and use linearized model instead of the nonlinear model. However for some plants assumption of linear model does not hold for long prediction horizons. Because the plant operating point changes for example based on some disturbances that act as a scheduling variable. An example could be a wind turbine for which wind speed acts as a scheduling variable and changes the operating point of the system.

2.1 Linear MPC formulation

The problem of linear MPC could be formulated as:

$$\begin{aligned}
\min_{u_0, u_1, \dots, u_{N-1}} \quad & \|x_N\|_{Q_f} + \sum_{i=0}^{N-1} \|x_{k+i|k}\|_Q + \|u_{k+i|k}\|_R \quad (6) \\
\text{Subject to} \quad & x_{k+1} = Ax_k + Bu_k \quad (7) \\
& u_{k+i|k} \in \mathbb{U} \quad (8) \\
& \hat{x}_{k+i|k} \in \mathbb{X} \quad (9)
\end{aligned}$$

Assuming that we use norms 1, 2 and ∞ the optimization problem becomes convex providing that the sets \mathbb{U} and \mathbb{X} are convex. Convexity of the optimization problem makes it tractable and guarantees that the solution is the global optimum. The problem above is based on a single linear model of the plant around one operating point. However below we formulate our problem using linear parameter varying systems (LPV) in which the scheduling variable is known for the entire prediction horizon.

2.2 Linear Parameter Varying systems

Linear Parameter Varying (LPV) systems are a class of linear systems whose parameters change based on a scheduling variable. Study of LPV systems was motivated by their use in gain-scheduling control of nonlinear systems (Apkarian et al. [1995]). LPV systems are able to handle changes in the dynamics of the system by parameter varying matrices.

Definition (LPV systems) let $k \in \mathbb{Z}$ denote discrete time. We define the following LPV systems:

$$x_{k+1} = A(\gamma_k)x_k + B(\gamma_k)u_k \quad (10)$$

$$A(\gamma_k) = \sum_{j=1}^{n_\gamma} A_j \gamma_{k,j} \quad B(\gamma_k) = \sum_{j=1}^{n_\gamma} B_j \gamma_{k,j} \quad (11)$$

Which $A(\gamma_k)$ and $B(\gamma_k)$ are functions of the scheduling variable γ_k . The variables $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$, and $\gamma_k \in \mathbb{R}^{n_\gamma}$ are the state, the control input and the scheduling variable respectively.

2.3 Problem formulation

Using the above definition, the linear parameter varying (LPV) model of the nonlinear system is of the following form:

$$\tilde{x}_{k+1} = A(\gamma_k)\tilde{x}_k + B(\gamma_k)\tilde{u}_k \quad (12)$$

This model is formulated based on deviations from the operating point. However we need the model to be formulated in absolute values of inputs and states. Because in our problem the steady state point changes as a function of the scheduling variable, we need to introduce a variable to capture its behavior. In order to rewrite the state space model in the absolute form we use:

$$\tilde{x}_k = x_k - x_k^* \quad (13)$$

$$\tilde{u}_k = u_k - u_k^* \quad (14)$$

x_k^* and u_k^* are values of states and inputs at the operating point. Therefore the LPV model becomes:

$$x_{k+1} = A(\gamma_k)(x_k - x_k^*) + B(\gamma_k)(u_k - u_k^*) + x_{k+1}^* \quad (15)$$

Which could be written as:

$$x_{k+1} = A(\gamma_k)x_k + B(\gamma_k)u_k + \lambda_k \quad (16)$$

with

$$\lambda_k = x_{k+1}^* - A(\gamma_k)x_k^* - B(\gamma_k)u_k^* \quad (17)$$

Now having the LPV model of the system we proceed to compute state predictions. In linear MPC predicted states at step n is:

$$\begin{aligned}
x_{k+n} &= A^n x_k + \sum_{i=0}^{n-1} A^i B u_{k+(n-1)-i} \quad (18) \\
&\text{for } n = 1, 2, \dots, N
\end{aligned}$$

However in our method the predicted state is also a function of scheduling variable $\Gamma_n = (\gamma_{k+1}, \gamma_{k+2}, \dots, \gamma_{k+n})^T$ for $n = 1, 2, \dots, N-1$ and we assume that the scheduling variable is known for the entire prediction. Therefore the predicted state could be written as:

$$x_{k+1}(\gamma_k) = A(\gamma_k)x_k + B(\gamma_k)u_k + \lambda_k \quad (19)$$

And for $n \in \mathbb{Z}, n \geq 1$:

$$\begin{aligned}
x_{k+n+1}(\Gamma_n) &= \left(\prod_{i=0}^n A^T(\gamma_{k+i}) x_k \right)^T \\
&+ \sum_{j=0}^{n-1} \left(\prod_{i=1}^{n-j} A^T(\gamma_{k+i}) \right)^T B(\gamma_{k+j}) u_{k+j} \quad (20) \\
&+ \sum_{j=0}^{n-1} \left(\prod_{i=0}^{n-j} A^T(\gamma_{k+i}) \right)^T \lambda_{k+(n-1)-j} \\
&+ B(\gamma_{k+n}) u_{k+n} + \lambda_{k+n}
\end{aligned}$$

Using the above formulas we write down the stacked predicted states which becomes:

$$X = \Phi(\Gamma)x_k + \mathcal{H}_u(\Gamma)U + \Phi_\lambda(\Gamma)\Lambda \quad (21)$$

with

$$X = (x_{k+1} \ x_{k+2} \ \dots \ x_{k+N})^T \quad (22)$$

$$U = (u_k \ u_{k+1} \ \dots \ u_{k+N-1})^T \quad (23)$$

$$\Gamma = (\gamma_k \ \gamma_{k+1} \ \dots \ \gamma_{k+N-1})^T \quad (24)$$

$$\Lambda = (\lambda_k \ \lambda_{k+1} \ \dots \ \lambda_{k+N-1})^T \quad (25)$$

In order to summarize formulas for matrices Φ, Φ_λ and \mathcal{H}_u , we define a new function as:

$$\psi(m, n) = \left(\prod_{i=n}^m A^T(\gamma_{k+i}) \right)^T \quad (26)$$

Therefore the matrices become:

$$\begin{aligned} \Phi(\Gamma) &= \begin{pmatrix} \psi(1, 1) \\ \psi(2, 1) \\ \psi(3, 1) \\ \vdots \\ \psi(N, 1) \end{pmatrix} \\ \Phi_\lambda(\Gamma) &= \begin{pmatrix} I & 0 & 0 & \dots & 0 \\ \psi(1, 1) & I & 0 & \dots & 0 \\ \psi(2, 1) & \psi(2, 2) & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi(N-1, 1) & \psi(N-1, 2) & \psi(N-1, 3) & \dots & I \end{pmatrix} \\ \mathcal{H}_u(\Gamma) &= \begin{pmatrix} B(\gamma_k) & 0 & \dots & 0 \\ \psi(1, 1)B(\gamma_k) & B(\gamma_{k+1}) & \dots & 0 \\ \psi(2, 1)B(\gamma_k) & \psi(2, 2)B(\gamma_{k+1}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \psi(N-1, 1)B(\gamma_k) & \psi(N-1, 2)B(\gamma_{k+1}) & \dots & B(\gamma_{N-1}) \end{pmatrix} \end{aligned}$$

After computing the state predictions as functions of control inputs, we can write down the optimization problem similar to a linear MPC problem as a quadratic program.

3. MINIMAX PROBLEM

3.1 Minimax for Linear Model

MPC uses a model of the system (to be controlled) to predict its future behavior. In nominal MPC the prediction of the state ($\hat{x}_{k+N|k}$) is a single value and it is calculated based on one model. However in robust MPC where the model is uncertain, this prediction is no longer a unique value, but it is a set instead. An approach to tackle the problem with uncertain model is to try to optimize the most pessimistic situation with respect to uncertainties. This means maximizing cost function on the uncertainty set. After maximization, we minimize the obtained cost function over control inputs as we do in nominal MPC. This approach is called minimax MPC which is a common solution to robust MPC problems (Löfberg [2003]). The special structure of our problem (having uncertainty only in the gain of the system) can help us simplifying the minimax MPC problem. Therefore we formulate robust MPC of our system in the form of minimax MPC of a system with uncertain gain (Löfberg [2003]):

$$x_{k+1} = Ax_k + B(\Delta_k)u_k \quad (27)$$

$$y_k = Cx_k + Du_k \quad (28)$$

We have employed norm-bounded uncertainty (Boyd et al. [1994]) to model our system:

$$B(\Delta_k) = B_0 + B_p \Delta_k C_p, \quad \Delta_k \in \Delta \quad (29)$$

$$\Delta = \{\Delta : \|\Delta\| \leq 1\} \quad (30)$$

And as the B matrices are dependent on γ , we have:

$$B(\gamma_k, \Delta_k) = B_0(\gamma_k) + B_p(\gamma_k) \Delta_k C_p(\gamma_k), \quad \Delta_k \in \Delta \quad (31)$$

$$\Delta = \{\Delta : \|\Delta\| \leq 1\} \quad (32)$$

With norm-bounded uncertain model of the system, we can formulate the minimax MPC with quadratic performance and soft constraints on inputs in the following form:

$$\begin{aligned} \min_u \max_{\Delta} \quad & \sum_{j=0}^{N-1} \|y_{k+j|k}\|_Q^2 + \|u_{k+j|k}\|_R^2 + \\ & \|v_{k+j|k}\|_{S_1}^2 + \|\xi_{k+j|k}\|_{S_2}^2 \\ \text{subject to} \quad & \hat{x}_{k+1|k} = A\hat{x}_{k|k} + B(\Delta_k)u_k \\ & \hat{y}_{k|k} = C\hat{x}_{k|k} + Du_k \\ & u_{k+j|k} \leq U_{max} + v_{k+j|k} \\ & u_{k+j|k} \geq U_{min} - v_{k+j|k} \\ & \Delta u_{k+j|k} \leq \Delta U_{max} + \xi_{k+j|k} \\ & \Delta u_{k+j|k} \geq \Delta U_{min} - \xi_{k+j|k} \\ & \eta_{k+j|k} \geq 0 \\ & \xi_{k+j|k} \geq 0 \end{aligned}$$

In order to simplify notations, we use stacked variables. The stacked output predictions, control sequences and auxiliary variables become:

$$U = (u_{k|k} \ u_{k+1|k} \ \dots \ u_{k+N-1|k})^T \quad (33)$$

$$\Delta U = (\Delta u_{k|k} \ \Delta u_{k+1|k} \ \dots \ \Delta u_{k+N-1|k})^T \quad (34)$$

$$Y = (\hat{y}_{k|k} \ \hat{y}_{k+1|k} \ \dots \ \hat{y}_{k+N-1|k})^T \quad (35)$$

$$\Xi = (\xi_{k|k} \ \xi_{k+1|k} \ \dots \ \xi_{k+N-1|k})^T \quad (36)$$

$$\Upsilon = (v_{k|k} \ v_{k+1|k} \ \dots \ v_{k+N-1|k})^T \quad (37)$$

$$\Phi_x(\Gamma) = (C\psi(1, 0) \ C\psi(1, 1) \ \dots \ C\psi(1, N-1))^T \quad (38)$$

Which gives:

$$\min_U \max_{\Delta^N} \quad Y^T Q Y + U^T R U + \Upsilon^T S_1 \Upsilon + \Xi^T S_2 \Xi \quad (39)$$

$$\text{subject to} \quad U \leq U_{max} + \Upsilon \quad (40)$$

$$U \geq U_{min} - \Upsilon \quad (41)$$

$$\Delta U \leq \Delta U_{max} + \Xi \quad (42)$$

$$\Delta U \geq \Delta U_{min} - \Xi \quad (43)$$

$$\Upsilon \geq 0 \quad (44)$$

$$\Xi \geq 0 \quad (45)$$

Where:

$$Y = \Phi_x(\Gamma) \hat{x}_{k|k-1} + \mathcal{H}_u(\Gamma) U + \Phi_\lambda(\Gamma) \Lambda \quad (46)$$

$$\mathcal{H}_u(\Gamma) = \mathcal{H}_u^0(\Gamma) + \mathcal{H}_u^\Delta(\Gamma, \Delta^N) \quad (47)$$

$$\Delta^N = (\Delta_1 \ \Delta_2 \ \dots \ \Delta_N)^T \quad (48)$$

And

$$\mathcal{H}_u^0(\Gamma) = \begin{pmatrix} CB(\gamma_k) & 0 & \dots & 0 \\ C\psi(1, 1)B(\gamma_k) & CB(\gamma_{k+1}) & \dots & 0 \\ C\psi(2, 1)B(\gamma_k) & C\psi(2, 2)B(\gamma_{k+1}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C\psi(N-1, 1)B(\gamma_k) & C\psi(N-1, 2)B(\gamma_{k+1}) & \dots & CB(\gamma_{N-1}) \end{pmatrix} \quad (49)$$

In above, Y is uncertain and depend on Δ^N . Our next task is to eliminate uncertainties from our optimization problem. To do so, we pull out the Δ variables. We start by writing the stacked output vector Y in the following form and then use theorem 1 to pull out the uncertainties:

$$\begin{aligned} Y &= \Phi_x(\Gamma) \hat{x}_{k|k-1} + \Phi_\lambda(\Gamma) \Lambda + \mathcal{H}_u^0(\Gamma) U \\ &\quad + \sum_{j=1}^N V_j \Delta_j W_j U, \quad \Delta_j \in \Delta \end{aligned} \quad (50)$$

$$\begin{aligned}
V_1 &= (CB_p(\gamma_k) \ C\psi(1,1)B_p(\gamma_k) \ \dots \ C\psi(N-1,1)B_p(\gamma_k))^T \\
V_2 &= (0 \ CB_p(\gamma_{k+1}) \ \dots \ C\psi(N-2,2)B_p(\gamma_{k+1}))^T \\
&\vdots \\
V_N &= (0 \ 0 \ 0 \ \dots \ CB_p(\gamma_{k+N-1}))^T \\
W_1 &= (C_p(\gamma_{k+1}) \ 0 \ \dots \ 0)^T \\
W_2 &= (0 \ C_p(\gamma_{k+1}) \ \dots \ 0)^T \\
&\vdots \\
W_N &= (0 \ 0 \ \dots \ C_p(\gamma_{k+1}))^T
\end{aligned}$$

Now we pull out the first uncertain element (Δ_1) from Y in the LMI constraint. To do so we define the following variable:

$$\begin{aligned}
\gamma_i &= \Phi_x(\Gamma)\hat{x}_{k|k-1} + \Phi_\lambda(\Gamma)\Lambda + \mathcal{H}_u^0(\Gamma)U \\
&+ \sum_{j=i}^N V_j \Delta_j W_j U, \quad i = 1, \dots, N
\end{aligned} \quad (51)$$

And afterwards we have:

$$\begin{aligned}
&\begin{pmatrix} t & \gamma_2^T & U^T & \Upsilon^T & \Xi^T \\ \star & Q^{-1} & 0 & 0 & 0 \\ \star & \star & R^{-1} & 0 & 0 \\ \star & \star & \star & S_1^{-1} & 0 \\ \star & \star & \star & \star & S_2^{-1} \end{pmatrix} + \begin{pmatrix} U^T W_1^T \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Delta_1^T \begin{pmatrix} 0 \\ V_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T \\
&+ \begin{pmatrix} 0 \\ V_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Delta_1 \begin{pmatrix} U^T W_1^T \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T \succeq 0
\end{aligned} \quad (52)$$

After pulling out the first uncertain element, we use the following theorem to find its equivalent certain LMI:

Theorem 1. robust satisfaction of the uncertain LMI:

$$F + L\Delta(I - D\Delta)^{-1}R + R^T(I - \Delta^T D^T)^{-1}\Delta^T L^T \succeq 0$$

is equivalent to the LMI

$$\begin{aligned}
\begin{bmatrix} F & L \\ L^T & 0 \end{bmatrix} \succeq \begin{bmatrix} R & D \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \tau I & 0 \\ 0 & -\tau I \end{bmatrix} \begin{bmatrix} R & D \\ 0 & I \end{bmatrix} \\
\tau \geq 0
\end{aligned}$$

Using theorem 1, it could be seen that the above LMI is equivalent to the following LMI:

$$\begin{aligned}
&\begin{pmatrix} t & \gamma_2^T & U^T & U^T W_1^T & \Upsilon^T & \Xi^T \\ \star & Q^{-1} & 0 & 0 & 0 & 0 \\ \star & \star & R^{-1} & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & S_1^{-1} & 0 \\ \star & \star & \star & \star & \star & S_2^{-1} \end{pmatrix} \\
&\succeq \begin{pmatrix} 0 & 0 \\ V_1 & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tau_1 I & 0 \\ 0 & -\tau_1 I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ V_1 & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{pmatrix}^T
\end{aligned} \quad (53)$$

Which is equivalent to:

$$\begin{pmatrix} t & \gamma_2^T & U^T & U^T W_1^T & \Upsilon^T & \Xi^T \\ \star & Q^{-1} - \tau_1 V_1 V_1^T & 0 & 0 & 0 & 0 \\ \star & \star & R^{-1} & 0 & 0 & 0 \\ \star & \star & \star & \tau_1 I & 0 & 0 \\ \star & \star & \star & \star & S_1^{-1} & 0 \\ \star & \star & \star & \star & \star & S_2^{-1} \end{pmatrix} \succeq 0 \quad (54)$$

We pulled out Δ_1 , and now we repeat the same procedure until we pull out all the uncertainties Δ_i for $i = 2, \dots, N$. Afterwards we apply Schur complement to write the final LMI in the form of smaller LMIs. Finally the optimization problem can be written in the following form:

$$\begin{aligned}
&\min_{t, \tau, U} \quad t_x + t_u + t_v + \sum_{j=0}^{N-1} t_j \\
&\text{subject to} \\
&\begin{pmatrix} t_x & \hat{x}_{k|k-1}^T \Phi_x(\Gamma)^T + \Phi_\lambda(\Gamma)\Lambda + U^T \mathcal{H}_u^0(\Gamma)^T \\ \star & Q^{-1} - \sum_{j=0}^{N-1} \tau_j V_j V_j^T \end{pmatrix} \succeq 0 \\
&\begin{pmatrix} t_u & U^T \\ \star & R^{-1} \end{pmatrix} \succeq 0 \quad \begin{pmatrix} t_v & \Upsilon^T \\ \star & S^{-1} \end{pmatrix} \succeq 0 \quad \begin{pmatrix} t_j & U^T W_j^T \\ \star & \tau_j I \end{pmatrix} \succeq 0 \\
&\begin{pmatrix} I \\ -I \\ \Psi \\ -\Psi \end{pmatrix} U - \begin{pmatrix} U_{max} + \Upsilon \\ -U_{min} + \Upsilon \\ \Delta U_{max} + I_0 u_{k-1} + \Xi \\ -\Delta U_{min} - I_0 u_{k-1} + \Xi \end{pmatrix} \leq 0 \\
&\tau_j \geq 0 \quad \Upsilon \geq 0 \quad \Xi \geq 0
\end{aligned} \quad (55)$$

We have used SeDuMi (Sturm [1999]) to solve this optimization problem. SeDuMi is a program that solves optimization problems with linear, quadratic and semidefinite constraints.

4. CASE STUDY

The case study here is a wind turbine. Wind turbine control is a challenging problem as the dynamics of the system change based on wind speed which has a stochastic nature. The method that we propose here is to use wind speed as a scheduling variable. With the advances in LIDAR technology (Harris et al. [2006]) it is possible to measure wind speed ahead of the turbine and this enables us to have the scheduling variable of the plant for the entire prediction horizon.

4.1 Modeling

In this section modeling of a wind turbine is explained. For detailed explanation on the modeling see (Mirzaei et al. [2012a]).

Nonlinear model For modeling purposes, the whole wind turbine can be divided into 4 subsystems: Aerodynamics subsystem, mechanical subsystem, electrical subsystem and actuator subsystem. To model the whole wind turbine, models of these subsystems are obtained and at the end they are connected together. The dominant dynamics of the wind turbine come from its flexible structure. Several degrees of freedom could be considered to model the flexible structure, but for control design mostly just a few important degrees of freedom are considered. In this work we only consider two degrees of freedom, namely the rotational degree of freedom (DOF) and drivetrain

torsion. Nonlinearity of the wind turbines mostly comes from its aerodynamics. Blade element momentum (BEM) theory (Hansen [2008]) is used to numerically calculate aerodynamic torque and thrust on the wind turbine. This theory explains how torque and thrust are related to wind speed, blade pitch angle and rotational speed of the rotor. Having aerodynamic torque and modeling drivetrain with a simple mass-spring-damper, the whole system equation with 2 degrees of freedom becomes:

$$J_r \dot{\omega}_r = Q_r - c(\omega_r - \frac{\omega_g}{N_g}) - k\psi \quad (56)$$

$$(N_g J_g) \dot{\omega}_g = c(\omega_r - \frac{\omega_g}{N_g}) + k\psi - N_g Q_g \quad (57)$$

$$\dot{\psi} = \omega_r - \frac{\omega_g}{N_g} \quad (58)$$

$$P_e = Q_g \omega_g \quad (59)$$

In which Q_r is aerodynamic torque, J_r and J_g are rotor and generator moments of inertia, ψ is the drivetrain torsion, c and k are the drivetrain damping and stiffness factors respectively lumped in the low speed side of the shaft and P_e is the generated electrical power. For numerical values of these parameters and other parameters given in this paper, we refer to (Jonkman et al. [2009]).

Linearized model To get a linear model of the system we need to linearize around some operating points. In our two DOFs model only the aerodynamic torque (Q_r) and electric power (P_e) are nonlinear. Taylor expansion is used to linearize them. Uncertainty in the measured wind speed and also in pitch actuator leads to uncertainty in the B matrix, yet the A matrix is known with enough accuracy. For more details on the uncertain state space model see (Mirzaei et al. [2012b]). Collecting all the discussed models, matrices of the state space model become:

$$A(\gamma) = \begin{pmatrix} \frac{a(\gamma) - c}{J_r} & \frac{c}{J_r} & -\frac{k}{J_r} \\ \frac{c}{N_g J_g} & -\frac{c}{N_g^2 J_g} & \frac{k}{N_g J_g} \\ 1 & -1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & Q_{g0} & 0 \end{pmatrix} \quad (60)$$

$$B(\gamma, \delta) = \begin{pmatrix} b_1(\gamma, \delta) & 0 \\ 0 & -\frac{1}{J_g} \\ 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \omega_{g0} \end{pmatrix} \quad (61)$$

In which $x = (\omega_r \ \omega_g \ \psi)^T$, $u = (\theta \ Q_g)^T$ and $y = (\omega_r \ \omega_g \ P_e)^T$ are states, inputs and outputs respectively. In the matrix B , parameter b_1 is uncertain. Therefore the uncertain linear state space model becomes:

$$\begin{aligned} \dot{x} &= A(\gamma)x + B(\gamma, \Delta)u \\ y &= Cx + Du \end{aligned}$$

4.2 Control objectives

The most basic control objective of a wind turbine is to maximize captured power during the life time of the wind turbine. This means trying to maximize captured power when wind speed is below its rated value. This is also called maximum power point tracking (MPPT). However when wind speed is above rated, control objective becomes

Table 1. Performance comparison between gain scheduling approach and linear MPC

Parameters	Proposed approach	Linear MPC
SD of ω_r (RPM)	0.042	0.103
SD of P_e (Watts)	4.158×10^4	9.975×10^4
Mean value of P_e (Watts)	4.998×10^6	4.998×10^6
SD of pitch (degrees)	2.781	3.005
SD of shaft moment (kNm)	295.26	482.49

regulation of the outputs around their rated values while trying to minimize dynamic loads on the structure. These objectives should be achieved against fluctuations in wind speed which acts as a disturbance to the system. In this work we have considered operation of the wind turbine in above rated (full load region). Therefore we try to regulate rotational speed and generated power around their rated values and remove the effect of wind speed fluctuations.

5. SIMULATIONS

In this section simulation results for the obtained controllers are presented. The controllers are implemented in MATLAB and are tested on a full complexity FAST (Jonkman and Jr. [2005]) model of the reference wind turbine (Jonkman et al. [2009]). Simulations are done with realistic turbulent wind speed, with Kaimal model (iec [2005]) as the turbulence model and TurbSim (Jonkman [2009]) is used to generate wind profile. In order to stay in the full load region, a realization of turbulent wind speed is used from category *C* of the turbulence categories of the IEC 61400-1 (iec [2005]) with $18m/s$ as the mean wind speed. Control inputs which are pitch reference θ_{in} and generator reaction torque reference Q_{in} along with system outputs which are rotor rotational speed ω_r and electrical power P_e are shown in figures 1-4. Sampling time and prediction horizon are chosen to be 0.1 and 10 respectively. Uncertainty is multiplicative and chosen to be %20 of the nominal value. Simulation results show good regulations of generated power and rotational speed. Table 1 shows a comparison of the results between the proposed approach and MPC with linearization at each sample point (Henriksen [2007]). As it could be seen from the table, the proposed approach gives better regulation on rotational speed and generated power (smaller standard deviations) than MPC, while keeping the shaft moment and pitch activity less. In all the figures 1-4 x-axis is time in seconds.

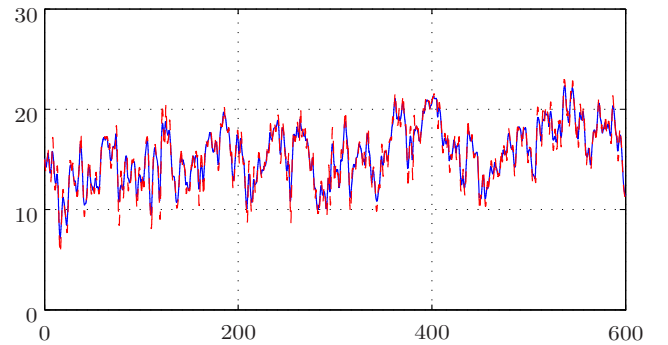


Fig. 1. Blade-pitch reference (degrees, red-dashed line is linear MPC and solid-blue line is the proposed approach)

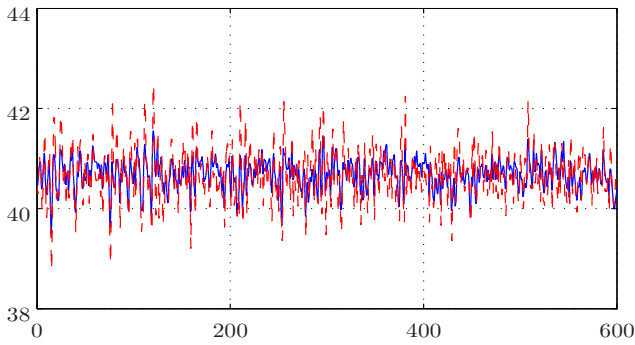


Fig. 2. Generator-torque reference (kNM, red-dashed line is linear MPC and solid-blue line is the proposed approach)

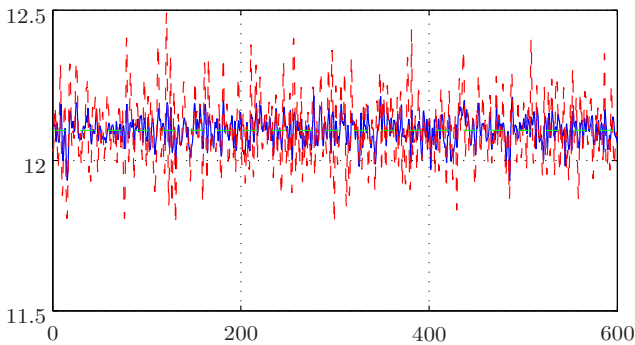


Fig. 3. Rotor rotational speed (ω_r , rpm, red-dashed line is linear MPC and solid-blue line is the proposed approach)

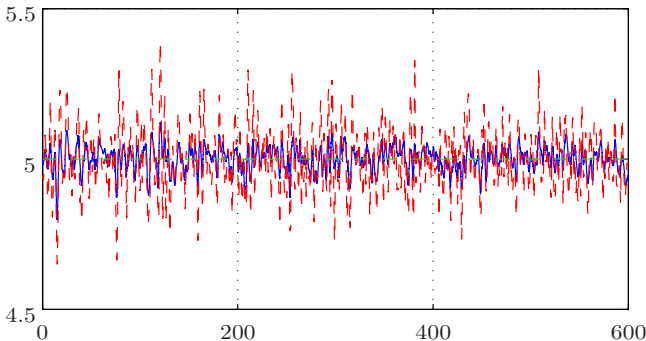


Fig. 4. Electrical power (mega watts, red-dashed line is linear MPC and solid-blue line is the proposed approach)

6. CONCLUSIONS

A method for dealing with robust MPC of nonlinear systems whose scheduling variable is known for the entire prediction horizon is proposed. The method is used for wind turbine control and the results are compared with a linear MPC that uses linearized model of the system at each sample time. Stability of the closed loop and recursive feasibility of the optimization problem are important issues that will be dealt with in future.

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